



Network Creation Games (a.k.a. Local Connection Games)





Introduction

- A **Local Connection Game (LCG)** is a game that models the *ex-novo* creation of a network
- Players are nodes that:
 - **Incur a cost** for the (adjacent) links they personally activate;
 - **Benefit** from having the other nodes on the network **as close as possible**, in terms of length of **shortest paths** on the created network (notice they can use all the activated edges)
- Introduced in: A. Fabrikant, A. Luthra, E. Maneva, C.H. Papadimitriou, S. Shenker, **On a network creation game**, PODC'03



The formal model

- n players: nodes $V=\{1,\dots,n\}$ in a graph to be built
- **Strategy** for player u : a set of incident edges (intuitively, a player buys these edges, that will be then used **bidirectionally** by everybody; however, only the owner of an edge can remove it, in case he decides to change its strategy)
- Given a strategy vector $S=(S_1,\dots, S_n)$, the **constructed network** will be the **undirected graph** $G(S)$
- player u 's **goal**:
 - to spend as little as possible for buying edges (**building cost**)
 - to make the distance to other nodes as small as possible (**usage cost**)

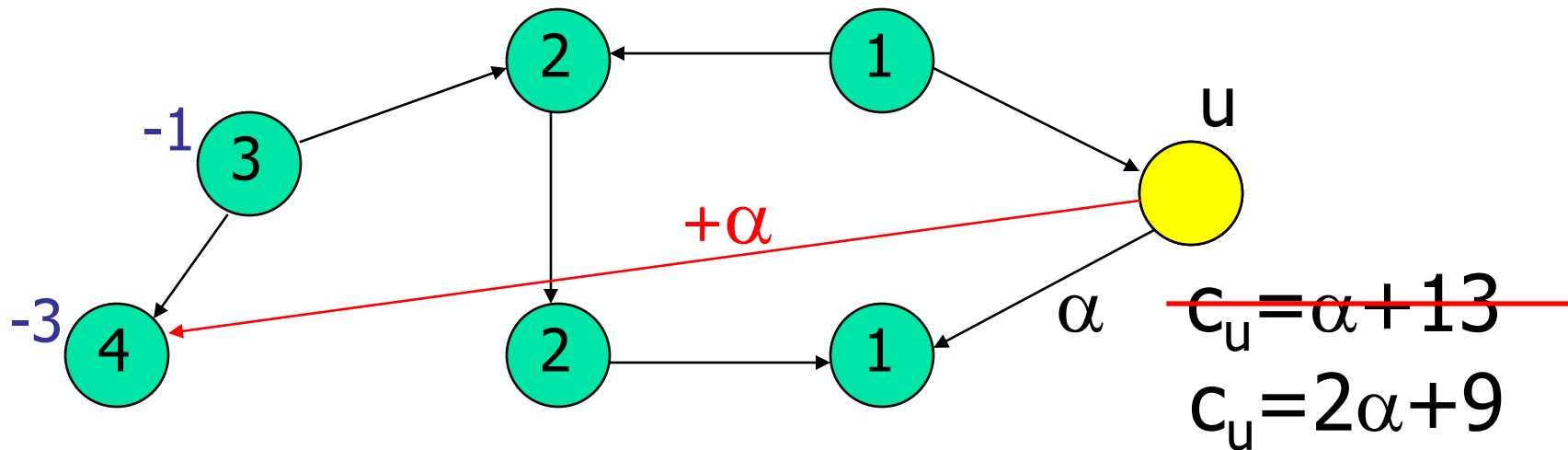


The model

- Each edge has a fixed real-value activation cost $\alpha \geq 0$
- $\text{dist}_{G(S)}(u,v)$: length of a shortest path (in terms of **number** of edges) in $G(S)$ between u and v
- n_u : number of edges bought by node u
- Player u aims to minimize its **cost**:

$$\text{cost}_u(S) = \alpha n_u + \sum_{v \in V} \text{dist}_{G(S)}(u,v)$$

Cost of a player: an example



Convention: arrow from the node buying the link

Notice that if $\alpha < 4$ this is an improving move for u



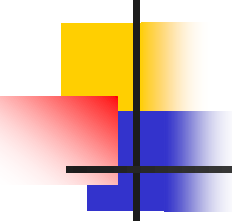
The social-choice function

- To evaluate the overall quality of a network, once again we consider the *utilitarian social cost*, i.e., the sum of all players' costs. Observe that:
 1. In $G(S)$ each term $\text{dist}_{G(S)}(u,v)$ contributes to the overall cost **twice**
 2. Each edge (u,v) is bought at most by one player

Social cost of a network $G(S)=(V,E)$:

$$SC(G(S)) = \alpha |E| + \sum_{u,v \in V} \text{dist}_{G(S)}(u,v)$$

Some (bad) computational aspects of LCG



- LCG are not **potential** games (differently from GCG); this can be shown by providing an instance in which a sequence of **improving** moves will generate a **cycle** in the corresponding space of strategy profiles
- As we will see, computing a **best-response** move for a player is **NP-hard** (differently from GCG)
- The complexity of establishing the existence of an **improving** move for a player (decision problem) is **open**
- The complexity of establishing the existence of a **NE** for a given α (decision problem) is **open**

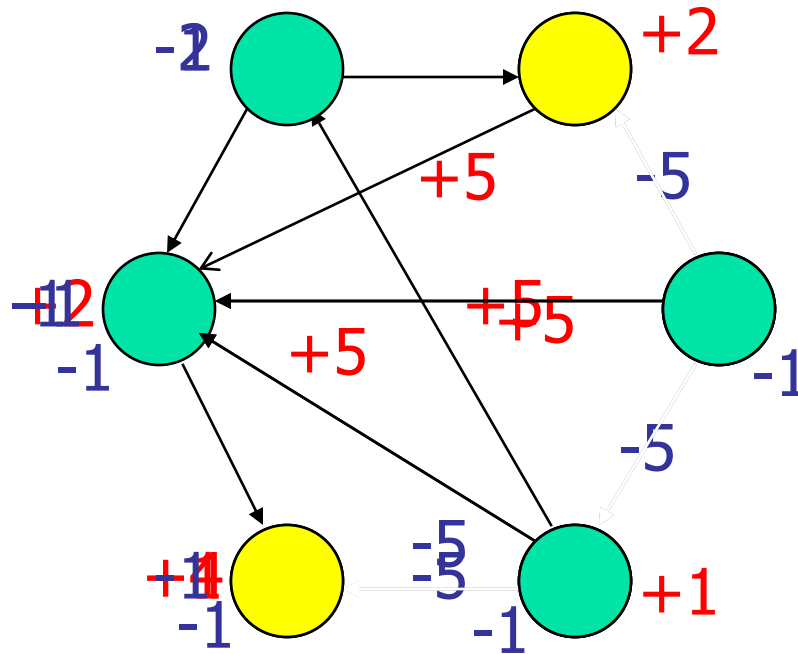


Our goal

- We use **Nash equilibrium** (NE) as the solution concept: Given a strategy profile S , the formed network $G(S)=(V,E)$ is **stable** (for the given value α) if S is a NE
- Conversely, given a graph $G=(V,E)$, it is **stable** if there exists a strategy vector S such that $G=G(S)$, and S is a NE
- Observe that any stable network must be **connected**, since the distance between two nodes is infinite whenever they are not connected
- A network is **optimal** or **socially efficient** if it minimizes the social cost
- We aim to characterize the **efficiency loss** resulting from **selfishness**, by bounding the **Price of Stability** (PoS) and the **Price of Anarchy** (PoA)

Stable networks: an example

- Set $\alpha=5$, and consider:



That's a stable network!

Theorem 1

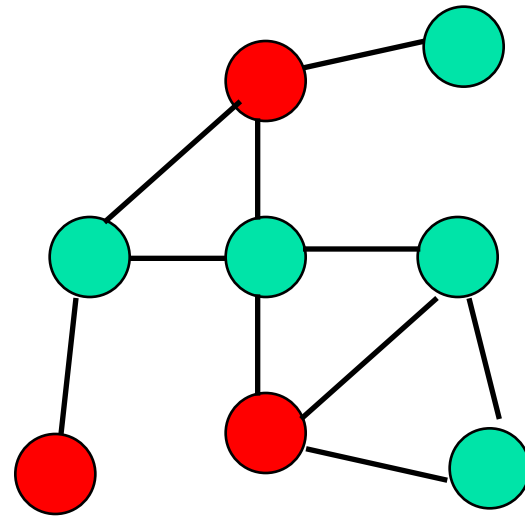
It is NP-hard, given the strategies of the other agents, to compute the best response of a given player in a LCG.

proof

Reduction from dominating set problem

Dominating Set (DS) problem

- Input:
 - a graph $G=(V,E)$
- Solution:
 - $U \subseteq V$, such that for every $v \in V-U$, there is $u \in U$ with $(u,v) \in E$
- Measure:
 - Cardinality of U



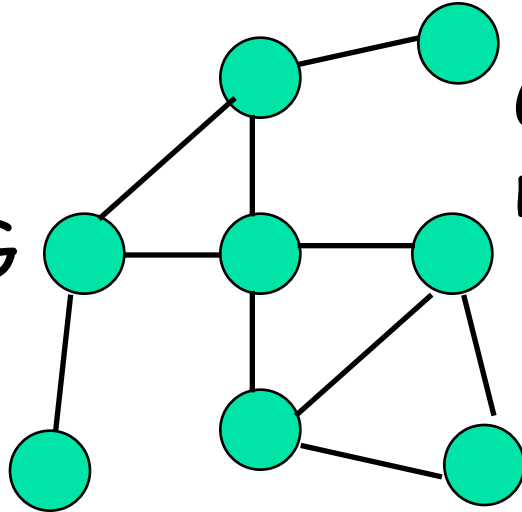
the reduction

$$1 < \alpha < 2$$

player i 

Instance of MDS

\Rightarrow Instance of LCG

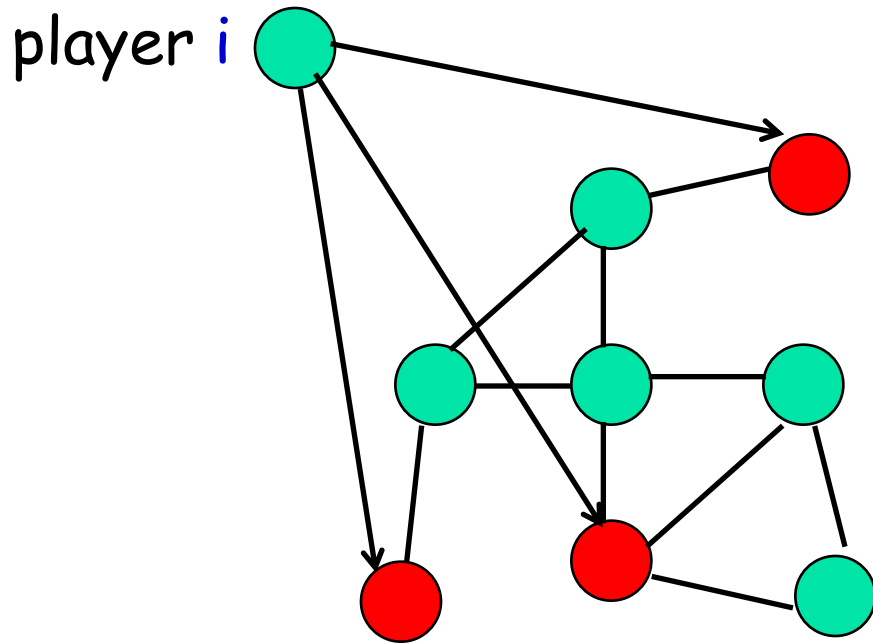


$$G=(V,E) = G(S_{-i})$$
$$n=|V| \quad S_{-i} \text{ is } S$$
$$\text{without } S_i$$

We will show that player i has a strategy yielding a cost $\leq \alpha k + 2n - k$ if and only if there is a DS of size $\leq k$

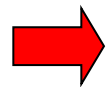
the reduction

$1 < \alpha < 2$



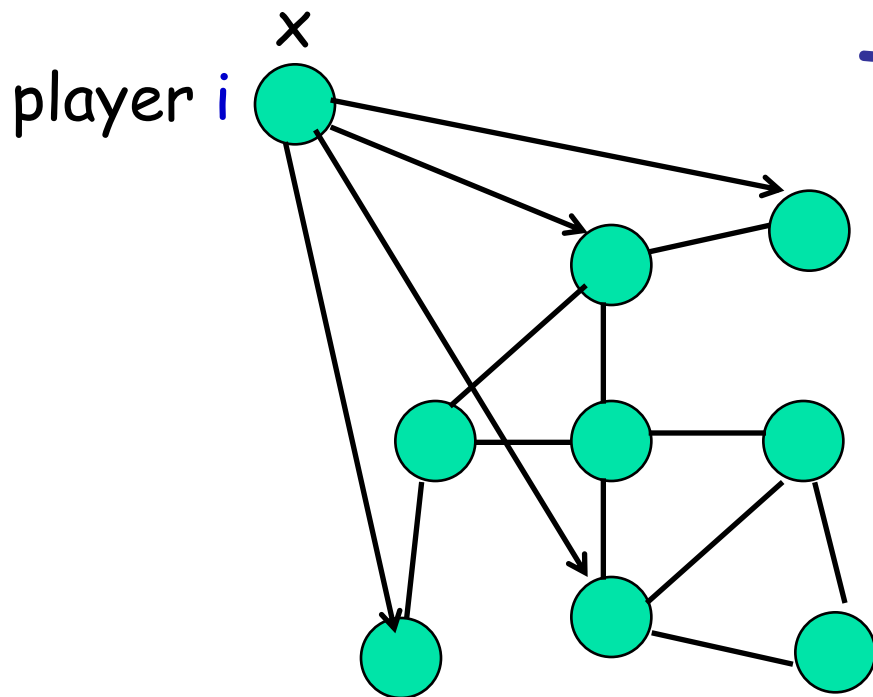
(\Leftarrow)

easy: given a dominating set U of size at most k in G , we want to show that there exists a strategy for player i costing at most $\alpha k + 2n - k$; then, let i buy edges incident to the nodes in U



Cost for i is $\alpha |U| + 2n - |U| = (\alpha - 1) |U| + 2n$
which is maximum for $|U| = k$, since $1 < \alpha < 2$,
i.e., it is $\leq (\alpha - 1) k + 2n$

$1 < \alpha < 2$



the reduction

$$G=(V,E) = G(S_{-i})$$

(\Rightarrow)

Let S_i be a strategy giving to player i a cost $cost_i(S) \leq \alpha k + 2n - k$

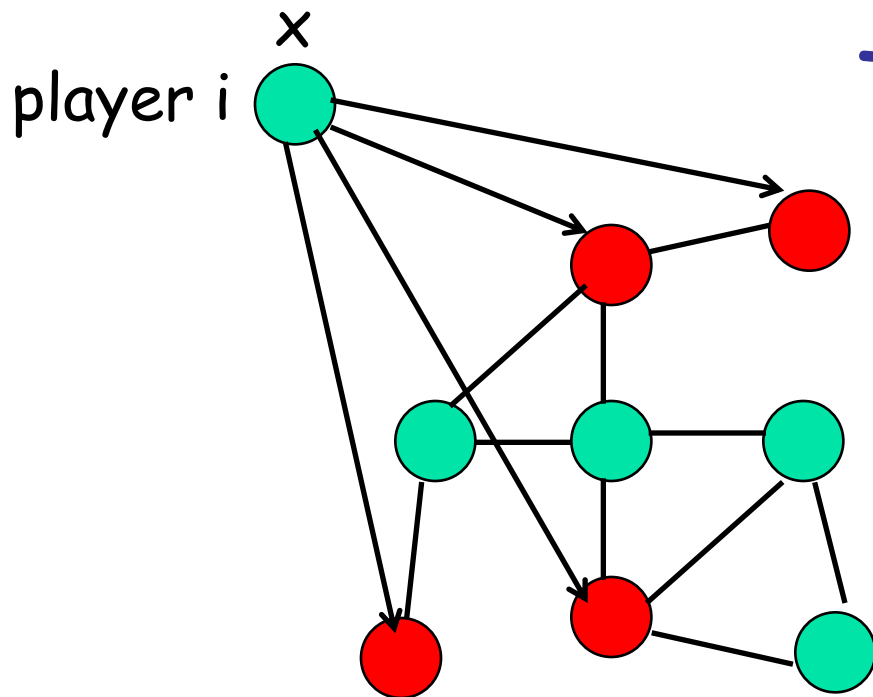
Modify S_i as follows:

repeat:

if there is a node v with distance ≥ 3 from x in $G(S)$, then add edge (x,v) to S_i (this decreases the cost of player i)

Finally, every node has distance either 1 or 2 from x , and the cost for i is still $\leq \alpha k + 2n - k$

$1 < \alpha < 2$



the reduction

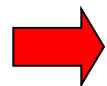
$$G=(V,E) = G(S_{-i})$$

(\Rightarrow)

Let U be the set of nodes at distance 1 from x ...

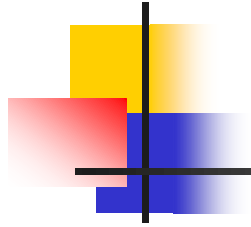
...it is easy to see that U is a dominating set of the original graph G

We have $\text{cost}_i(S) = \alpha|U| + 2n - |U|$, and moreover $\text{cost}_i(S) \leq \alpha k + 2n - k$



$$(\alpha-1)|U| \leq (\alpha-1)k \text{ and since } \alpha > 1 \\ |U| \leq k$$



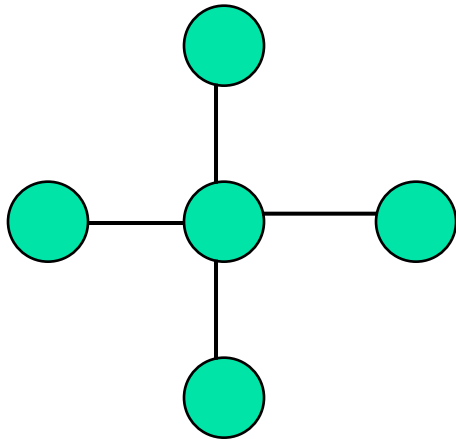
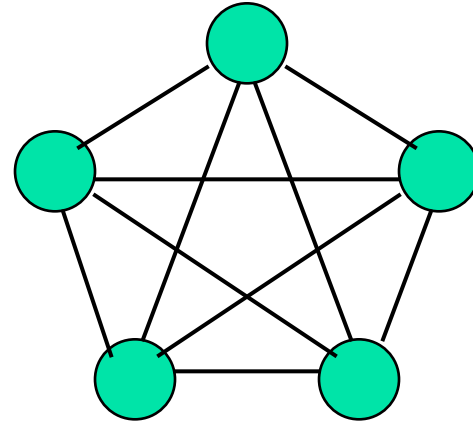


*How does an optimal
network look like?*



Some notation

K_n : complete graph
with n nodes



A **star** is a tree
of height at most 1
(when rooted at its
center)

Lemma 1

If $\alpha \leq 2$ then the complete graph is an optimal solution, while if $\alpha \geq 2$ then the star is an optimal solution.

proof

Let $G=(V,E)$ be an optimal solution; $|E|=m$ and $SC(G)=OPT$

$$\begin{aligned} OPT &= \alpha |E| + \sum_{u,v \in V} \text{dist}_G(u,v) \geq \alpha m + \color{red}{2m} + \color{blue}{2(n(n-1) - 2m)} \\ &= (\alpha - 2)m + 2n(n-1) \longleftarrow \color{blue}{LB(m)} \quad \begin{array}{l} \color{red}{\text{adjacent nodes}} \\ \color{red}{\text{at distance 1}} \end{array} \quad \begin{array}{l} \color{blue}{\text{non-adjacent pairs of}} \\ \color{blue}{\text{nodes at distance } \geq 2} \end{array} \end{aligned}$$

Notice: $LB(m)$ is equal to $SC(K_n)$ when $m = n(n-1)/2$, and to $SC(\text{star})$ when $m = n-1$; indeed:

$$SC(K_n) = \alpha n(n-1)/2 + n(n-1)$$

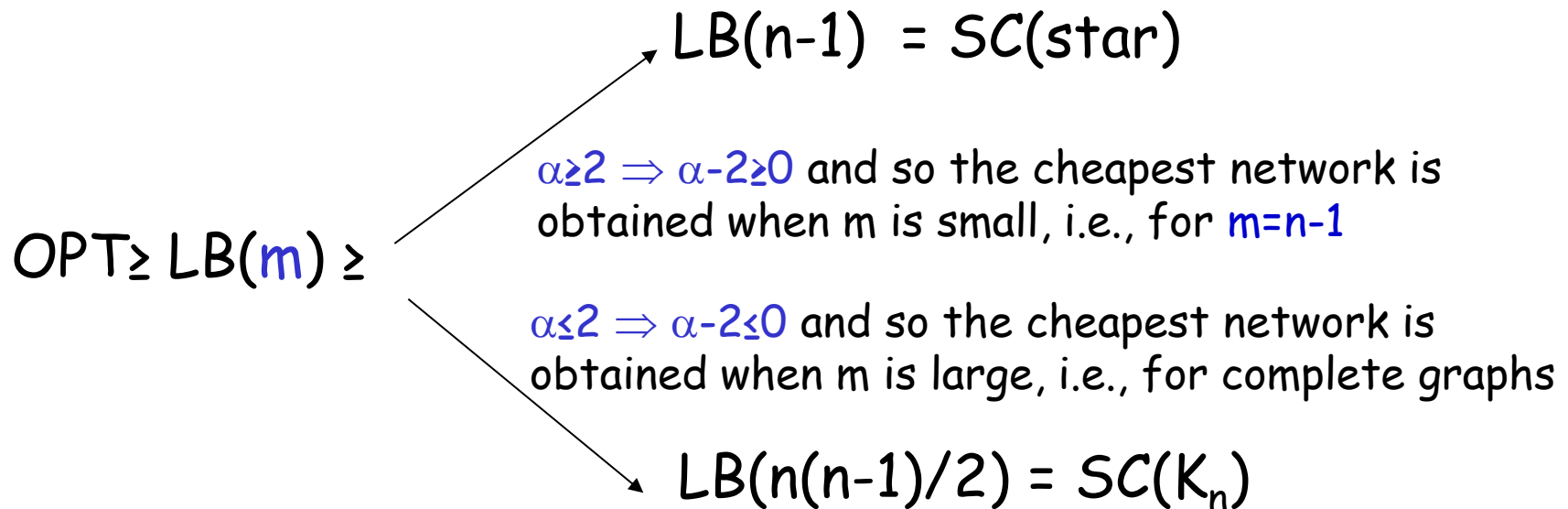
$$SC(\text{star}) = \alpha (n-1) + 2(n-1) + 2(n-1)(n-2) = \alpha (n-1) + 2(n-1)^2$$

and it is easy to see that they correspond to $LB(n(n-1)/2)$ and to $LB(n-1)$, respectively.

Proof (continued)

$G=(V,E)$: optimal solution;
 $|E|=m$ and $SC(G)=OPT$

$$LB(m) = (\alpha - 2)m + 2n(n-1)$$





*Are complete graphs
and stars stable?*

Lemma 2

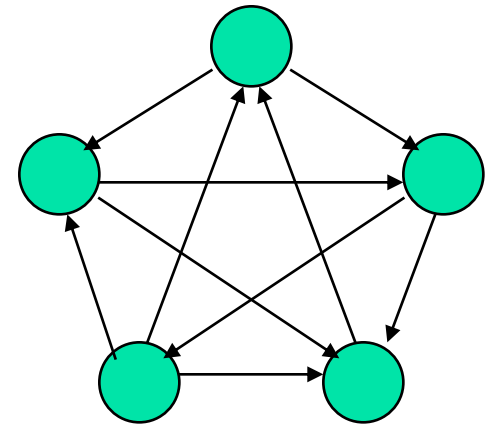
If $\alpha \leq 1$ the complete graph is stable, while if $\alpha \geq 1$ then the star is stable.

Proof:

$\alpha \leq 1$

By definition, we have to find a NE S inducing a clique. Actually, any arbitrary strategy profile S inducing a clique is a NE.

Indeed, if a node removes any $k \geq 1$ owned edges, it saves αk in the building cost, but it pays $k \geq \alpha k$ more in the usage cost (the k detached nodes are now at distance 2)

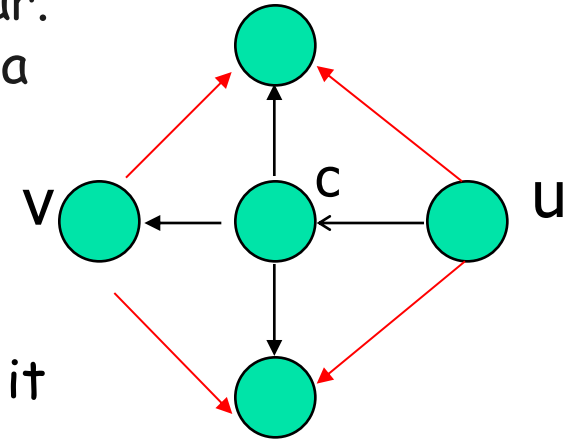


Proof (continued) $\alpha \geq 1$

By definition, we have to find a NE S inducing a star. Actually, any arbitrary strategy profile S inducing a star is a NE. Indeed:

Center c cannot change its strategy, otherwise its cost increase to infinity

If a leaf v not buying edges buys any $1 \leq k \leq n-2$ edges it pays αk more in the building cost, but it saves only $k \leq \alpha k$ in the usage cost



For a leaf u buying an edge, its cost is $\alpha + 1 + 2(n-2)$ and we have two cases:
Case 1: u maintains (u, c) and buys any $1 \leq k \leq n-2$ additional edges; this case is similar to the previous one.

Case 2: u removes (u, c) and buys any $1 \leq k \leq n-2$ edges; thus, it pays αk in the building cost, and its usage cost becomes $k + 2 + 3(n-k-2)$, and so its total cost becomes:

distance to distance to distance to
adjacent nodes c other nodes

$$\alpha k + k + 2 + 3n - 3k - 6 = \alpha + [\alpha(k-1) - 2k + n] + 2(n-2) \geq \quad (\text{since } \alpha \geq 1)$$

$$\alpha + [k-1-2k+n] + 2(n-2) = \alpha + [n-k-1] + 2(n-2)$$

which is at least equal to the initial cost of $\alpha + 1 + 2(n-2)$, since the quantity in square brackets is at least 1, being $1 \leq k \leq n-2$. ■

Theorem 2

For $\alpha \leq 1$ and $\alpha \geq 2$ the PoS is 1. For $1 < \alpha < 2$ the PoS is at most $4/3$

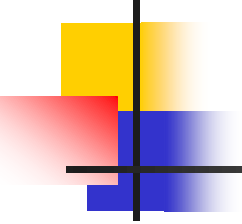
Proof: From Lemma 1 and 2, for $\alpha \leq 1$ (respectively, $\alpha \geq 2$) a **complete graph** (respectively, a **star**) is both optimal and stable, and so the claim follows.

$1 < \alpha < 2$ K_n is an optimal solution (Lemma 1), and a star T is stable (Lemma 2); then

$$\text{PoS} \leq \frac{SC(T)}{SC(K_n)} = \frac{\alpha(n-1) + 2(n-1)^2}{\alpha n(n-1)/2 + n(n-1)} < \frac{2n(n-1)}{n(n-1)/2 + n(n-1)} = 4/3$$

$< 2(n-1)$ for $1 < \alpha < 2$
 $> n(n-1)/2 + n(n-1)$ for $1 < \alpha < 2$





What about the Price of Anarchy?

...for $\alpha < 1$ the complete graph is the
only stable network,
(try to prove that formally)
hence $PoA=1$...

...for larger value of α ?



State-of-the-art

$\alpha = 0$	1	2	$\sqrt[3]{n/2}$	$\sqrt{n/2}$	$O(n^{1-\epsilon})$	$273n$	$12n \lg n$	∞
PoA	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{\mathcal{O}(\sqrt{\log n})}$	< 5	≤ 1.5

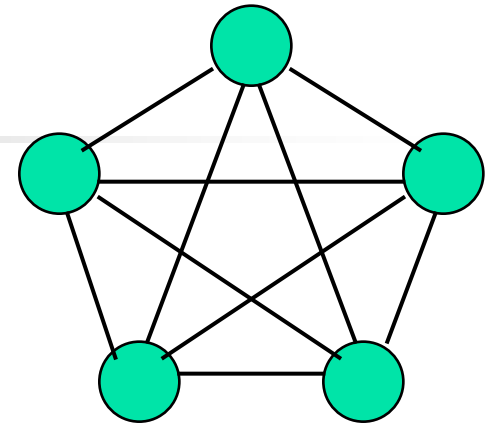
Many of these results are quite technical; we will show a simpler bound, namely that

$$PoA = O(\sqrt{\alpha})$$

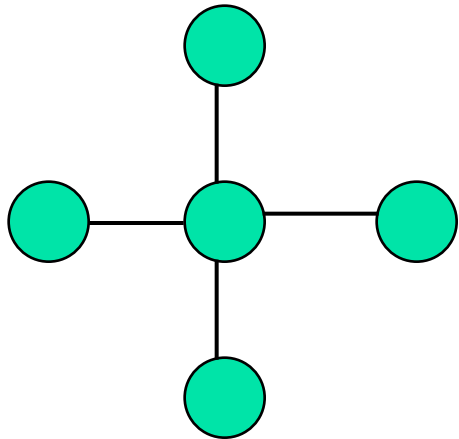


Some more notation

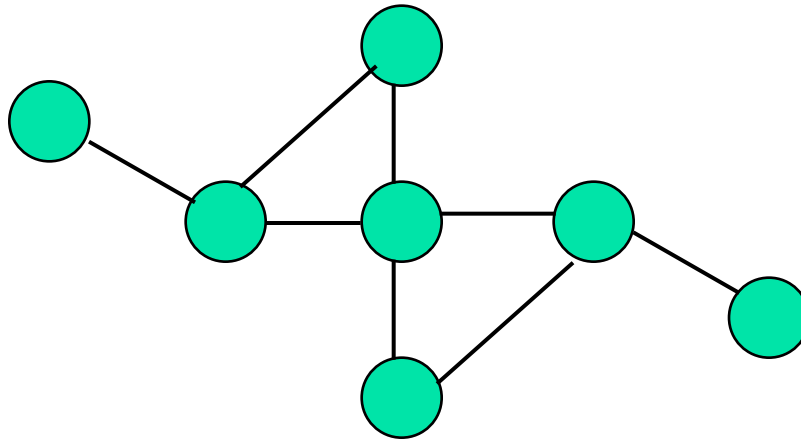
The **diameter** of a graph G is the maximum distance between any two nodes



diam=1



diam=2

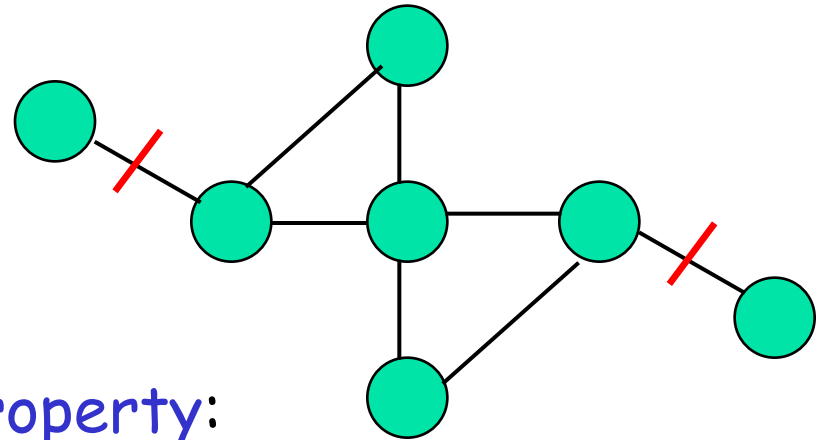


diam=4

Some more notation

An edge e is a **cut edge** (a.k.a. **bridge**) of a graph $G=(V,E)$ if $G-e$ is disconnected

$$G-e=(V,E\setminus\{e\})$$



A simple property:

Any graph has at most $n-1$ cut edges (indeed, if we take any spanning tree T of G , all the non-tree edges cannot be bridges, and T has exactly $n-1$ edges (not all of them are bridges, clearly))

Theorem 3

The PoA of the LCG is at most $6\sqrt{\alpha} + 3$.

proof

It follows from the following lemmas that we are going to prove:

Lemma 3

The diameter of any stable network is at most $2\sqrt{\alpha} + 1$.

Lemma 4

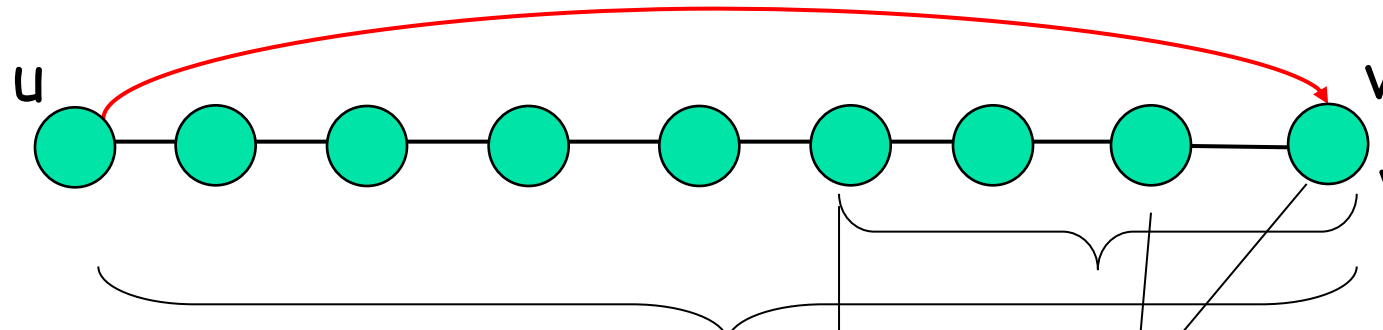
The SC of any stable network with diameter d is at most $3d$ times the optimum SC.

Indeed, from these lemmas it follows that the SC of any stable network is at most $(6\sqrt{\alpha} + 3) \cdot \text{SC}(\text{OPT})$, and so the PoA is at most $6\sqrt{\alpha} + 3$.

proof of Lemma 3 Let G be a stable network.

Consider a shortest path in G between two nodes u and v .

Assume its length is even, i.e., equal to $2k$ for some k . The case in which it is odd is similar. Then:



If u buys (u,v) k vertices reduce their distance from u

$\text{dist}_G(u,v) = 2k$
for some k (in the example $k=4$)

from $2k$ to 1	\rightarrow	$2k-1$
from $2k-1$ to 2	\rightarrow	$2k-3$
\vdots		
from $k+1$ to k	\rightarrow	1

saving in distance

...since G is stable:

$\rightarrow \alpha \geq k^2 \quad \rightarrow k \leq \sqrt{\alpha}$

$$\sum_{i=0}^{k-1} (2i+1) = k^2$$

$\rightarrow \text{dist}_G(u,v) \leq 2\sqrt{\alpha} + 1$



To prove **Lemma 4** we will make use of the following:

Proposition 1

Let G be a network with diameter d , and let $e=(u,v)$ be a non-cut edge. Then in $G-e$, every node w increases its distance from u by at most $2d$

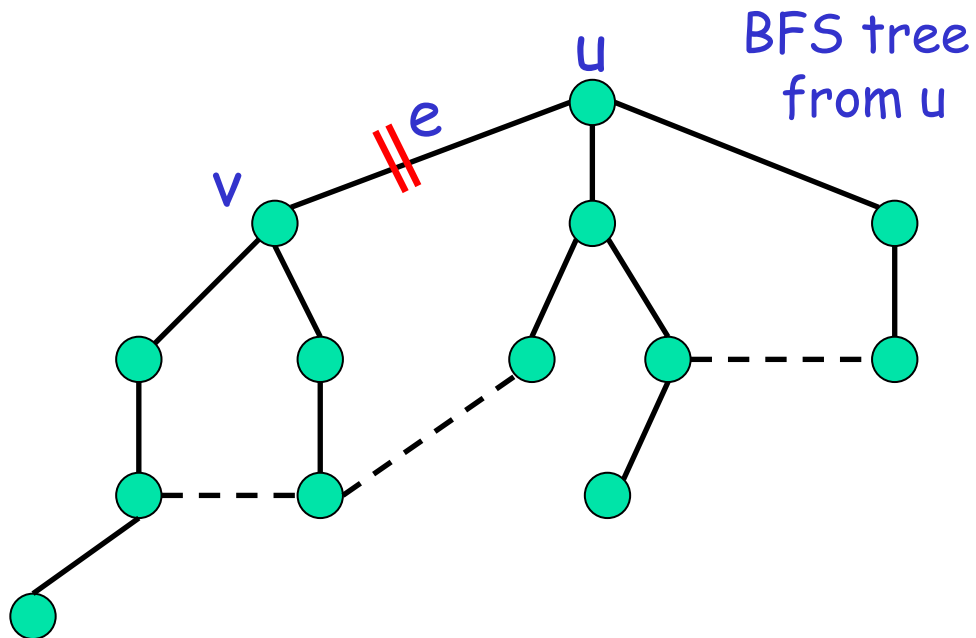
Proposition 2

Let G be a stable network, and let F be the set of non-cut edges bought by a node u . Then $|F| \leq (n-1)2d/\alpha$

Proposition 1

Let G be a network with diameter d , and let $e=(u,v)$ be a non-cut edge. Then in $G-e$, every node w increases its distance from u by at most $2d$

proof

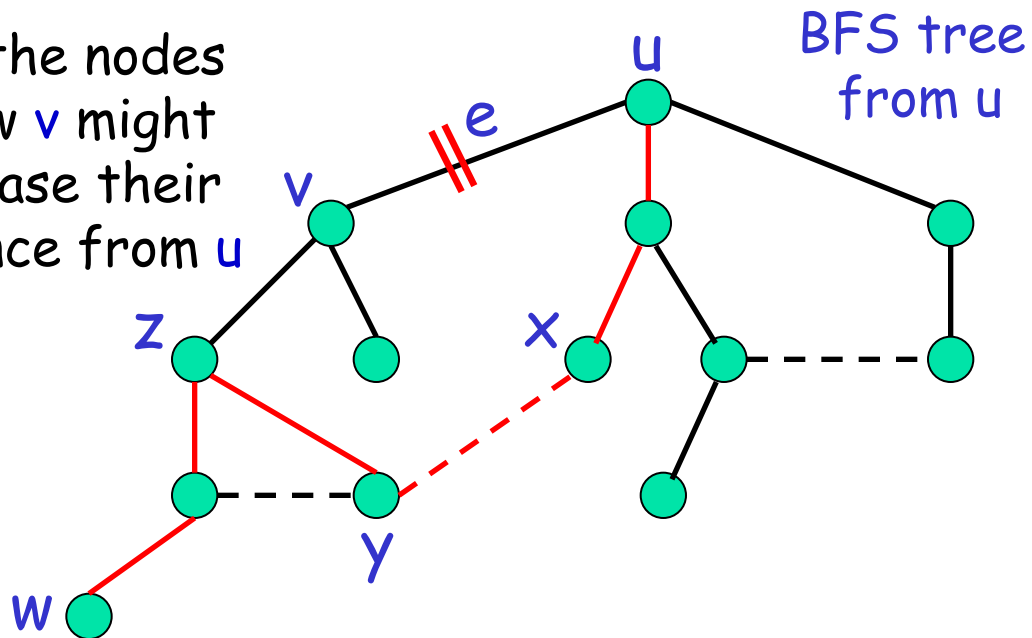


Proposition 1

Let G be a network with diameter d , and let $e=(u,v)$ be a non-cut edge. Then in $G-e$, every node w increases its distance from u by at most $2d$

proof

only the nodes below v might increase their distance from u



(x,y) :
any edge crossing the cut induced by the removal of e

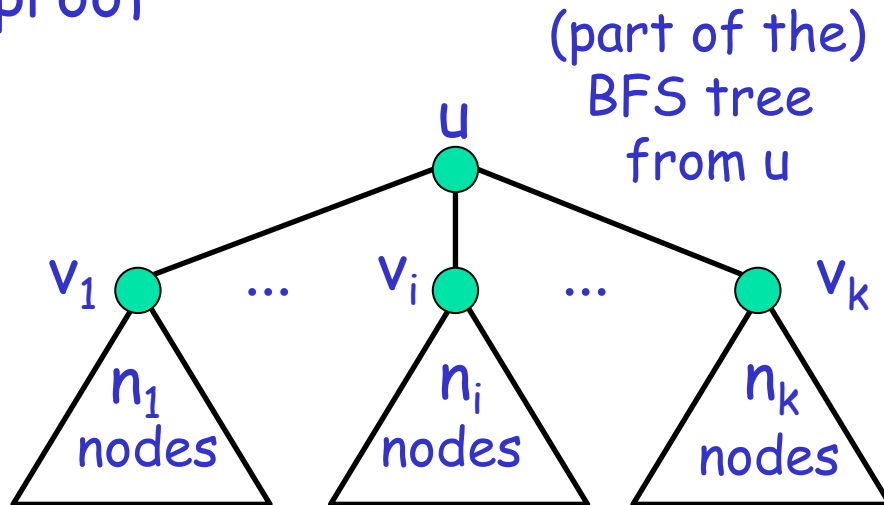
$$d_{G-e}(u,w) \leq \underbrace{d_G(u,x)}_{\leq d} + 1 + \underbrace{d_G(y,z)}_{\leq d} + \underbrace{d_G(z,w)}_{\leq d_G(u,w)-1} \leq d_G(u,w) + 2d$$



Proposition 2

Let G be a stable network, and let F be the set of non-cut edges bought by a node u . Then $|F| \leq (n-1)2d/\alpha$

proof



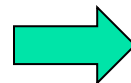
$$k = |F|$$

if u removes (u, v_i) it saves α and its distance cost increases by at most $2d n_i$ (Prop. 1)

since G is stable:
 $\alpha \leq 2d n_i$

by summing up for all i

$$k \alpha \leq 2d \sum_{i=1}^k n_i \leq 2d (n-1)$$



$$k \leq (n-1) 2d/\alpha$$



Lemma 4

The SC of any stable network $G=(V,E)$ with diameter d is at most $3d$ times the optimum SC.

proof

$OPT \geq \alpha (n-1) + n(n-1)$ [notice this is the building cost of a star and the usage cost of a clique!]

$$SC(G) = \underbrace{\sum_{u,v} d_G(u,v)}_{\leq dn(n-1)} + \alpha |E| \leq d OPT + 2d OPT = 3d OPT$$

$$\alpha |E| = \underbrace{\alpha |E_{\text{cut}}|}_{\leq (n-1)} + \underbrace{\alpha |E_{\text{non-cut}}|}_{\leq n(n-1)2d/\alpha} \leq \alpha(n-1) + n(n-1)2d \leq 2d OPT$$

Prop. 2

